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№ 1

$$\begin{cases} u_t = a^2 \Delta_2 u & 0 \le r \le 1 \quad 0 \le \varphi < 2\pi \\ u|_{t=0} = r\sin\varphi \\ u|_{r=1} = 0 \\ |u(0)| < \infty \end{cases}$$

Решение:

• $u(r, \varphi, t) = T(t)\cdot W(r, \varphi)$

$T' W = a^2 T \Delta_2 W$

$$\frac{\Delta_2 W}{W} = \frac{T'}{a^2 T} = -\lambda$$

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$$\begin{cases} \Delta_2 W = -\lambda W \\ W|_{r=0} = 0 \\ |W|_{r=0}| < \infty \end{cases}$$

• $W = R(r)\cdot \Phi(\varphi) \Rightarrow$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}(R\Phi)\right) + \frac{1}{r^2}\frac{\partial^2 (R\Phi)}{\partial \varphi^2} = -\lambda R\Phi$$

$$\frac{1}{r}(R'' r + R')\Phi + \frac{1}{r^2} R\Phi'' = -R\Phi$$

$$\eta = \frac{R'' r^2 + R' r}{R} + \lambda r^2 + \underbrace{\frac{\Phi''}{\Phi}}_{=-\eta} = 0$$

• $\begin{cases} \Phi'' = -\eta\Phi \\ \Phi(\varphi) = \Phi(\varphi + 2\pi) \end{cases}$

$\eta_0 = 0 \quad \Phi_0 = 1$

$\eta_n = n^2 \quad \Phi_n = A_n \cos n\varphi + B_n \sin n\varphi$

$n \in \mathbb{N}$

• $R'' r^2 + R' r + R(\lambda r^2 - \eta) = 0$

$\eta_n = n^2$

Замена $x = \sqrt{\lambda}\, r$

$R'' x^2 + R' x + R(x^2 - n^2) = 0$

$R_n(x) = C_n J_n(x) + D_n N_n(x)$

$D_n = 0$, т.к N_n ведет себя плохо в 0

$R_n(1) = 0 = C_n J_n(\sqrt{\lambda}\, 1)$

$\Rightarrow \sqrt{\lambda} = \mu_{nk} \quad \lambda_{nk} = (\mu_{nk})^2$

• $u(r, \varphi, t) = \sum\limits_{n=0}^{\infty}\sum\limits_{k=1}^{\infty} J_n(\mu_{nk} r)\cdot$

$\cdot (A_{nk}\cos n\varphi + B_{nk}\sin n\varphi)\, e^{-(\mu_{nk} a)^2 t}$

$t = 0 \Rightarrow$

$$r\sin\varphi = \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} J_n(\mu_{nk} r)(A_{nk}\cos n\varphi + B_{nk}\sin n\varphi)$$

$A_{nk} = 0 \quad B_{nk} = 0$ кроме $n = 1$

$$B_{1k} = \frac{2}{(J_1'(\mu_{1k}))^2}\int_0^1 r\cdot r\cdot J_1(\mu_{1k} r)\, dr =$$

$$= \left\{ \begin{matrix} \text{замена} \\ \mu_{1k} r = s \end{matrix} \right\} = \frac{2}{(J_1'(\mu_{1k}))^2}\int_0^{\mu_{1k}} \frac{s^2}{\mu_{1k}^3} J_1(s)\, ds$$

$$= \frac{2\mu_{1k}^2 J_2(\mu_{1k})}{(J_1'(\mu_{1k}))^2 (\mu_{1k})^3} = \frac{-2}{\mu_{1k} J_1'(\mu_{1k})}$$

$$\boxed{u(r, \varphi, t) = \sum_{k=1}^{\infty} J_1(\mu_{1k} r)\sin\varphi \cdot e^{-(\mu_{1k} a)^2 t}\cdot\left(\frac{-2}{\mu_{1k} J_1'(\mu_{1k})}\right)}$$

№2 2/2

$$\begin{cases} u_t = a^2 \Delta_3 u \\ u\big|_{t=0} = J\left(\frac{\mu_0^{(5)}}{b} r\right) \sin \frac{2\pi z}{h} \\ u\big|_{r} = 0 \end{cases}$$

круглый цилиндр радиуса b и высоты h

Решение

Условия не зависят от φ, будем искать ответ в виде $u = u(r, z, t)$

- $u(r,z,t) = W(r,z)\cdot T(t)$

  $\frac{\Delta_2 W(r,z)}{W(r,z)} = \frac{T'(t)}{a^2 T(t)} = -\lambda$

- $\Delta_3 W(r,z) = -\lambda W$

  $W(r,z) = R(r)\cdot Z(z)$

  $\frac{1}{r}(rR'' + R')Z + RZ'' + \lambda RZ = 0$

  $\underbrace{\frac{1}{r}\frac{(rR''+R')}{R}}_{=\eta} + \lambda + \underbrace{\frac{Z''}{Z}}_{=-\alpha} = 0$

- $Z: \begin{cases} Z'' = -\alpha Z \\ Z(0) = 0 \\ Z(h) = 0 \end{cases}$

  $Z = A\cos(\sqrt{\alpha} z) + B\sin(\sqrt{\alpha} z)$

  $A = 0 \qquad \sqrt{\alpha}\cdot h = \pi n$

  $\Rightarrow \alpha_n = \left(\frac{\pi n}{h}\right)^2 \quad n \in \mathbb{N}$

- $\frac{1}{r}\left(\frac{rR''+R'}{R}\right) = \eta$

  $r^2 R'' + R'r - \eta r^2 R = 0$

  Замена $\sqrt{\eta} r = x$

  $x^2 R'' + xR' - x^2 R = 0$

  $R(x) = A\cdot J_0(x) + BN_0(x)$

  $B = 0$, т.к N плохо в 0

  $R(\sqrt{\eta} b) = 0 \Rightarrow \eta_k = \left(\frac{\mu_0^k}{b}\right)^2 \quad k \in \mathbb{N}$

- $\lambda_{nk} = \eta_n + \alpha_k = \left(\frac{\mu_0^n}{b}\right)^2 + \left(\frac{\pi k}{h}\right)^2$

- $u(r,z,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A_{nk} J_0\left(\frac{\mu_0^k r}{b}\right)\sin\left(\frac{\pi n z}{h}\right) \cdot e^{-a^2\lambda_{nk} t}$

- $u(r,z,0) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} A_{nk} J_0\left(\frac{\mu_0^k r}{b}\right)\sin\frac{\pi n z}{h} = J\left(\frac{\mu_0^{(5)} r}{b}\right)\sin\left(\frac{2\pi z}{h}\right)$

  $\Rightarrow A_{52} = 1$, остальные 0

$$\Rightarrow \boxed{u(r,z,t) = J_0\left(\frac{\mu_0^{(5)} r}{b}\right)\sin\frac{2\pi z}{h}\cdot e^{-a^2\left(\left(\frac{\mu_0^{(5)}}{b}\right)^2 + \left(\frac{2\pi}{h}\right)^2\right)t}}$$